ASYMPTOTIC DIMENSION OF FINITELY PRESENTED GROUPS

THANOS GENTIMIS

Abstract. We prove that if a finitely presented group is one-ended then its asymptotic dimension is bigger than 1. It follows that a finitely presented group of asymptotic dimension 1 is virtually free.

1. Introduction

The notion of asymptotic dimension of a metric space was introduced by Gromov in [5]. It is a large scale analog of topological dimension and it is invariant by quasi-isometries. This notion has proved relevant in the context of Novikov's higher signature conjecture and it was investigated further by other people (see [10], [1], [8]).

In this paper we show the following:

Theorem 1. *If G is a one-ended finitely presented group then G has asymptotic dimension greater or equal to 2.*

Also we deduce as corollary:

Theorem 2. *If G is a finitely presented group with asdim $G = 1$ then G is virtually free.*

For finitely generated groups the statement above doesn't hold. We give a counter-example at the end.

After we completed this work T.Januszkiewicz brought to our attention his joint paper with J.Swiatkowski ([6]) where they proved the same results independently.

Finally I would like to thank pr. P.Papasoglu for his help and guidance during the writing of this paper.

2. Preliminaries

Metric Spaces. Let (X, d) be a metric space. If A, B are subsets of X we set $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. A *path* in X is a map $\gamma : I \to X$ where I is an interval in $\mathbb{R}$. A path γ joins two points x and y in X if $I = [a, b]$, $\gamma(a) = x$ and $\gamma(b) = y$. The path γ is called an *infinite ray* starting from x_0 if $I = [0, \infty)$ and $\gamma(0) = x_0$. A *geodesic*, a *geodesic ray*, or a *geodesic segment* in X, is an isometry $\gamma : I \to X$ where I is $\mathbb{R}$ or $[0, \infty)$ or a closed interval in $\mathbb{R}$. We use the terms geodesic, geodesic ray etc. for the images of γ without discrimination. On a path-connected space X given two points x, y we define the *path metric* to

Date: September 13, 2006.

Key words and phrases. dimension theory, asymptotic dimension, finitely presented group.

Research co-funded by the European Social Fund and National Resources-EPEAEK II-PYTHAGORAS.

be $\rho(x,y) = inf\{length(p)\}$ where the infimum is taken over all paths p that join x and y. A space is called a *geodesic metric space* if for every x, y in X there exists a geodesic segment which joins them. In a geodesic space the path metric is indeed a metric. A geodesic metric space X is said to be *one-ended* if for every bounded K, $X - K$ has exactly one unbounded connected component. We say that X is *uniformly one-ended*, if for every $n \in \mathbb{R}^+$ there is an $m \in \mathbb{R}^+$ such that for every $K \subset X$ with $diam(K) < n$, $X - K$ has exactly one connected component of diameter bigger than m.

Groups. The *Cayley graph* of G with respect to a generating set S is the 1-dimensional complex having a vertex for each element of G and an edge joining vertex x to vertex xs for every vertex x and every $s \in S$. The Cayley graph has a natural metric which makes it a geodesic metric space, where each edge has length 1 (see [8]). In fact any connected graph can be made geodesic metric space in the same way.

We will use the same letter G for both the group and its Cayley graph as a metric space. We also use *van - Kampen Diagrams* (see [7] chapter V pg 236 -240 and [3]). A van Kampen diagram $\mathfrak{D}$ for a word w in S representing the identity element of G, is a finite, planar, contractible, combinatorial 2-complex; its 1-cells are directed and labeled by generators and the boundary labels of each of its 2-cells are cyclic conjugates of relators or inverse relators. Further the boundary label for $\mathfrak{D}$ is w when read (by convention anticlockwise) from a base point in $\partial\mathfrak{D}$. We recall here that a word w represents the identity element of G if and only if the path in the Cayley graph labeled by w is closed.

There is a natural map f from the 1-skeleton $\mathfrak{D}^{(1)}$ of $\mathfrak{D}$ to the Cayley graph of G. f sends the base point to a vertex v of the Cayley graph and edges of $\mathfrak{D}^{(1)}$ to edges of the Cayley graph with the same label. Obviously f is determined by the image of the base point, v. f is not necessarily injective. If we consider $\mathfrak{D}^{(1)}$ as a geodesic metric space giving each edge length 1 then $d(x,y) \geq d(f(x), f(y))$ for every x, y in $\mathfrak{D}^{(1)}$.

Finally we say that a group G is *virtually free* if there exists a finite index subgroup H of G which is a free group.

A group G is called free-by-infinite if it contains a normal subgroup N of finite index such that N is free.

Generally if X is a property of groups then we say that G is virtually X if $\exists\, H <_f G$ and H has property X. We say that G is X-by-infinite if $\exists\, N \lhd_f G$ normal such that N has property X.

Obviously if G is X-by-infinite then G is virtually X. The converce also holds if the property X is inherited to subgroups.

Now since if G is free every subgroup of G is free we have that G is virtually free if and only if G is free by-infinite.

Asymptotic Dimension. A metric space Y is said to be d - disconnected or that it has dimension 0 on the d - scale if there exist $B_i \subset Y$ such that:

$$Y \; = \bigcup_{i \in I} B_i$$

with $\sup\{diamB_i,\ i \in I\} \leq D < \infty$, $\mathrm{d}(B_i, B_j) \geq d \ \forall\ i \neq j$.

Definition. *(Asymptotic Dimension 1) We say that a space X has asymptotic dimension n if n is the minimal number such that for every $d > 0$ we have:*
$X = \bigcup X_k$ for $k = 1, 2, ..., n$ and all X_k are d-disconnected. We then write $asdim\, X = n$.

We say that a covering $\{B_i\}$ of X has *d - multiplicity k*, if and only if every d-ball $B(x, d)$ in X meets no more than k sets B_i of the covering. A covering has *multiplicity n* if no more than $n + 1$ sets of the covering have a non empty intersection. A covering $\{B_i\},\ i \in I$ is *D - bounded*, if $diam(B_i) \leq D\, \forall\, i \in I$.

Definition. *(Asymptotic Dimension 2) We say that a space X has $asdimX = n$, if n is the minimal number such that $\forall\, d > 0$ there exists a D-bounded covering of X with d - multiplicity $\leq n + 1$.*

The two definitions used here are the first two definitions Gromov gave in his paper [5]. It is not difficult to see that the two definitions are equivalent.

3. Main Theorem

Before we get to the main theorem we will prove two lemmas that we will need below.

Lemma 1. *Let G be a finitely generated, infinite group then asdim $G > 0$.*

Proof. Let asdim $G = 0$ and fix $d > 0$. Then according to the first definition we have that $G = X_1$ were $X_1 = \bigcup B_i$ with:

(1) $diamB_i \leq D$, $\forall\, i \in I$
(2) $d(B_i, B_j) \geq d$, $\forall\, i \neq j$.

That means that G is d-disconnected. Since G is a connected graph it follows that we can not have two distinct B_i's. So $G \subset B_1$ which means that G is D-bounded. But since G is finitely generated we have immediately that G is finite which is a contradiction. □

Lemma 2. *If G is an one-ended finitelly presented group then G contains a bi-infinite geodesic.*

Proof. Let fix x_0 the vertex that corresponds to the identity element in the Cayley graph of the group. Since G is infinite and connected, for every $n \in \mathbb{N}$ there exists a vertex x_n with distance n from e. We denote by G_n all the geodesics from e to x_n. Then define $X = \bigcup G_n$. Obivously X is an infinite set. Thus there exists an edge that starts from x_0 let it be e_1 such that infinite geodesics pass from that e_1. Let x_1 be the other end of e_1 then there exists an e_2 that starts from x_1 such that infinite geodesics pass through e_2. We continue this way and we get an infinite geodesic. Remark that this infinite ray is not the only one but the procedure gives as at least one.

So we have obtained one infinite ray r. Since we are working with the Cayley graph of the group for every $n \in N$ there exists an element of G, lets denote it h_n, such that $h_n \cdot r_n = x_0$ namely h_n takes the $n - th$ vertex of r to the x_0 and $\|h_n\|$ is the least possible. Lets denote by $h_n \cdot r$ the path that we take if we apply the element h_n to every vertex of r. Then since our metric is G - invariant we have that all $h_n \cdot r$ are infinite geodesics rays starting from $h_h \cdot x_0$ respectively. Lets denote by Y all of these geodesics. Obviously Y is infinite, thus becides e_1 there exists another edge e_1', starting from x_0 such that infinite geodesic rays of Y pass from e'_1. Let x'_1 be the other edge of e'_1 then there exists an edge e'_2 that starts from x'_1 such that infinite geodesics pass through e'_2. We continue this way and we get an infinite path, streching towards the other side. We use that and we have our bi-infinite geodesic. □

Theorem 1. *If G is an one-ended finitely presented group then asdim $G \geq 2$.*

Proof. By lemma 1 and since G is one-ended we have that G is infinite and thus asdim $G > 0$ We will show that asdim $G \neq 1$. Let's suppose that asdim $G = 1$.

Let $M = \max\{|r_i|, i = 1, 2, ..., n\colon\ r_i$ relation of $G\}$, where $|r| = length$ of the word r. We fix $d > 100M + 100$. Since asdim $G = 1$ there is a covering $\mathbb{B} = \{B_i\}$ with:

$$G = \bigcup_{i \in I} B_i$$

and $diam B_i < D$, $\forall\, i \in I$, such that every ball $B(x, d)$ intersects at most 2 sets of the covering $\mathbb{B}$. We may assume without loss of generality that if r is a path in the Cayley graph labeled by a relator r_i then r is contained in some $B_j \in \mathbb{B}$.

Since G is one-ended we have that G has a bi-infinite geodesic S(Lemma 2). Let $N = max\{100D^{100}, 300M\}$. Choose an $x_0 \in S$ and consider the ball $B(x_0, N)$ which separates the geodesic S into two geodesic rays S_1 and S_2. Since G is one-ended there is an x in S_1, a y in S_2 and a path p with $p(0) = x$ and $p(t) = y$ such that $p \bigcap B(x_0, N) = \emptyset$.

We denote by $[x, y]$ the part of the geodesic S, that connects x and y. Obviously $length([x, y]) \geq 2N$. We denote by w the path that corresponds to $[x, y] \bigcup p$
We have then that

$$length(w) = length([x, y]) + length(p) \Rightarrow length(w) > 200D$$

So in order to cover the path w we need at least 3 sets of the covering $\{B_i\}$.
We consider now the van-Kampen diagram $\mathfrak{D}$ that corresponds to the path w and the function f from $\mathfrak{D}^{(1)}$ to the Cayley graph G. So $f(\partial\mathfrak{D}) = w$. For notational convenience we label vertices and edges of $\partial\mathfrak{D}$ in the same way as w. So for example we denote the vertex on $\partial\mathfrak{D}$ which is mapped to $x_0 \in w$ by f also by x_0.

Let B be a set of the covering that intersects $[x, y]$. We consider $f^{-1}(B)$. Let $C(B)$ be the union of all 2-cells of $\mathfrak{D}$ which have the property that their boundary is contained in $f^{-1}(B)$. Let U be the collection of all such sets B with the following property: For some connected component, K, of $C(B)$, $[x, y] \cap K$ is contained in an interval $[a, b]$ with $a, b \in K$ such that $x_0 \in [a, b]$. Let $d(K) = d(a, b)$ for such a component and let $d(B)$ be the maximal value of all $d(K)$ for K component of $C(B)$ such that $x_0 \in [a, b]$. Let B_1 be a set in U for which $d(B_1)$ is maximal. Let K_1 be the connected component of $C(B_1)$ for which $d(K_1) = d(B_1)$. Let's say that $K_1 \cap [x, y]$ is contained in $[a_1, b_1]$ with $a_1, b_1 \in K_1$. Let e be the edge of $[x, y]$ adjacent to a_1 which does not lie in $[a, b]$. If r is the 2-cell containing e there is some $B_2 \in \mathbb{B}$ such that $C(B_2)$ contains r.

Let C be the subset of $\mathfrak{D}$ which contains all 2-cells with boundary contained in $f^{-1}(B_1 \bigcup B_2)$. Since $d(x_0, p) \geq N$, C does not intersect p. Thus $\mathfrak{D} - C \neq \emptyset$. Let K be the connected component of C which contains x_0.

Let $P = \overline{\mathfrak{D} - K} \bigcap \overline{K}$. P is connected since K is connected. Each edge of P is contained in two 2-cells. One of these 2-cells lies in $f^{-1}(B_1 \bigcup B_2)$ and one does not lie in this set. It is not possible that all edges of P are contained in a 2-cell of $f^{-1}(B_2)$. Indeed in this case we would have $d(B_2) > d(B_1)$, which is impossible. Since r is contained in $C(B_2)$ some edge of P is not contained in a 2-cell of $f^{-1}(B_1)$. It follows that there are 2 adjacent edges e_1, e_2 in P such that one of them is contained in a 2-cell of $f^{-1}(B_1)$ and the other in a 2-cell of $f^{-1}(B_2)$. If c is the 2-cell that contains e_1 and is not in $f^{-1}(B_1 \bigcup B_2)$ then c lies in a set $f^{-1}(B_3)$ with $B_3 \in \mathbb{B}$ and $B_3 \neq B_1, B_2$. The edges e_1, e_2 and the 2-cell c have a vertex v in common. So $v \in f^{-1}(B_1 \cap B_2 \cap B_3)$. It follows that $B_1 \cap B_2 \cap B_3 \neq \emptyset$, a contradiction.

This concludes the proof. □

Remark. *The result above holds also for uniformly one-ended simply connected simplicial complexes. So if X is a uniformly one-ended simply connected simplicial complex then asdim $X \geq 2$.*

One can prove this in a similar way and we give a sketch here: Since X is one-ended then X has geodesic segments $[a_n, b_n]$ of length n for every $n \in \mathbb{N}$. Using these segments and the fact that X is uniformly one ended we can construct a closed path in X like the path w in the previous proof. Finally since X is simply connected there is a map $f : D \to X$ with $f(\partial D) = w$, where D is a disc. We may further assume that f is simplicial by the simplicial approximation theorem. So this disc D replaces the van-Kampen diagram $\mathfrak{D}$ is the previous proof.

Of course the result does not hold for one-ended simply connected simplicial complexes, a half-line gives a counterexample. We remark finally that if a Cayley graph is one ended then it is uniformly one ended.

We note that the following theorem holds:

Theorem. *(Dunwoody-Stallings [2]) If G is a finitely presented group then G is the fundamental group of a graph of groups such that all the edge groups are finite and all the vertex groups are 0 or 1 ended.*

Also it is known that:

Lemma 3. *If all the vertex groups are 0-ended (i.e. finite) then G is virtually free (see [9], page 120, prop.11).*

Furthermore it is not difficult to prove the following lemma (see [4]):

Lemma 4. *If $H < G$ and H is finitely generated then asdim $G \geq$ asdim H.*

Using the lemma above and theorem 1 we have the stronger result:

Theorem 2. *If G is a finitely presented group with asdim $G = 1$ then G is virtually free.*

Proof. Let G be a finitely presented group with asdim $G = 1$. Let Γ be the graph of groups of the Dunwoody-Stallings theorem. If a vertex group H is one-ended then from the theorem 1, asdim $H \geq 2$. But $H < G$ which means that asdim $G \geq 2$ which is a contradiction. So all vertex groups are 0-ended. It follows that G is virtually free. □

Now we give an example of a finitely generated group which is not finitely presented, not virtually free and that has asymptotic dimension 1. Namely:

Proposition. *Let $G = \mathbb{Z}_2 \wr_{wr} \mathbb{Z}$ the wreath product of $\mathbb{Z}_2$ and $\mathbb{Z}$. Then asdim $G = 1$ and G is not virtually free.*

Proof. Since $G = \mathbb{Z}_2 \wr_{wr} \mathbb{Z}$ there exists a short exact sequence:

$$0 \to (\oplus_{\mathbb{Z}}\mathbb{Z}_2) \to G \to \mathbb{Z} \to 0$$

By the Hurewicz type formula (see [11]) we have:

$$asdim G \leq asdim\mathbb{Z} + asdim(\oplus_{\mathbb{Z}}\mathbb{Z}_2)$$

Since every finitely generated subgroup F of $(\oplus_{\mathbb{Z}}\mathbb{Z}_2)$ is finite, we have that all F of that type have asymptotic dimension 0. Following the definition of asymptotic dimension for arbitrary discrete groups found in [11] we get:

$$asdim(\oplus_{\mathbb{Z}}\mathbb{Z}_2) = \sup\{\text{asdim}F | F < G, \text{finitely generated}\} = 0$$

Another way to get the same result is by using the following corollary found in [12]

Corollary. *Let G be a countable abelian group. Then asdim $G = 0$ if and only if G is torsion.*

Obviously $\oplus_{\mathbb{Z}}\mathbb{Z}_2$ is abelian and torsion so asdim $\oplus_{\mathbb{Z}}\mathbb{Z}_2 = 0$. Thus we get:

$$\text{asdim } G \leq 1 + 0 = 1$$

Since G is a finitely generated infinite group (see [13] for the description of the generators), by lemma 1 we have that asdim $G > 0$. So asdim $G = 1$.

We will prove that G is not virtually free. Let G be virtually free. Then G is free-by-infinite. So there exists a normal subgroup N of G such that N is free and the index $|G : N|$ is finite. Recall the exact sequence:

$$0 \to (\oplus_{\mathbb{Z}}\mathbb{Z}_2) \to G \to \mathbb{Z} \to 0$$

and lets denote the image of $\oplus_{\mathbb{Z}}\mathbb{Z}_2$ in G to be H through the mapping f. Then obviously $H < G$ and we will prove that $N \bigcap H = \{e\}$. That is true since if $x \in N \bigcap H$ we will have that $x \in H$ and thus there exists a y in $\oplus_{\mathbb{Z}}\mathbb{Z}_2$ with $f(y) = x$. But since $o(y) = 2$ or 1 we have that $y^2 = e$ in $\oplus_{\mathbb{Z}}\mathbb{Z}_2$ which gives us that $f(y)^2 = e$ in G, meaning $x^2 = e$ in G. But x belongs to N and N is free so we must have that $x = e$. Now from the second theorem of isomorphisms and since N is a normal subgroup of G we have:

$$\frac{H}{H \bigcup N} \simeq \frac{H \cdot N}{N}$$

but $H \bigcup N = \{e\}$ and $\frac{H \cdot N}{N} < \frac{G}{N}$ thus we get:

$$H \simeq \frac{H}{H \bigcup N} \simeq \frac{H \cdot N}{N} < \frac{G}{N}$$

But that leads to a contradiction since H is infinite (f is one to one and $\oplus_{\mathbb{Z}}\mathbb{Z}_2$ is infinite) whereas $\frac{G}{N}$ is finite. This concludes the proof. □

References

1. A.Dranishnikov, *Asymptotic topology*, Russian Math.Surveys 55(2000), No 6, 71-116.
2. M.Dunwoody, *The accessibility of finitely presented groups*, Invent. Math. 81 (1985), No 3, 449-457.
3. S.Gersten and Tim Riley, *Filling length in Finitely presentable groups*, Geometriae Dedicata 92: 4158, 2002.
4. M.Gromov, *Hyperbolic groups*, Essays in group theory (S. M. Gersten, ed.), MSRI Publ. 8, Springer-Verlag, 1987 pp. 75-263.

5. M.Gromov, *Asymptotic invariants of infinite groups*, 'Geometric group theory', (G.Niblo, M.Roller, Eds.), LMS Lecture Notes, vol. 182, Cambridge Univ. Press (1993)
6. T.Januszkiewicz and J.Swiatkowski, *Filling Invariants In Systolic Complexes and Groups* "preprint" (June 2005).
7. R.Lyndon, P.Schupp, *Combinatorial Group Theory*, Springer - Verlag (1970)
8. J.Roe, *Lectures On Coarse Geometry*, Univ. Lect. Series Vol. 31 Amer.Math.Society.(2003)
9. J.P.Serre, *Trees*, Springer-Verlag (1980)
10. G.Yu, *The Novicov conjecture for groups with finite asymptotic dimension*, Ann. of Math. 147(1998), No 2, 325-335.
11. A.Dranishnikov and J.Smith,*Asymptotic Dimension of Discrete Groups*, Fundamenta Mathematicae, 189 (2006)
12. J.Smith,*On Asymptotic Dimension Of Countable Abelian Groups*, Topology and Applications, Vol 153,No 12,2047, June (2006)
13. A.Dranishnikov,*Groups With Polynomial Dimension Growth*, Geometrica Dedicata, arxiv: math.MG/0405239

Mathematics Department, University of Athens, Athens 157 84, Greece
E-mail address: gentimisth@math.uoa.gr